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Analysis of Piece-wise Linear Systems by the Method of Integral Equations

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by

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Abstract

Analysis of piece-wise linear systems may require the solution of high order linear differential equations whose parameters are constants within a given region but change into different constants for adjacent regions. The multiple regions of such a system may be identified with discrete intervals and it is a simple matter to obtain the system response by the method of integral equations. These solutions are given in the form of convergent infinite series, the terms of which may be easily evaluated by a digital computer. The time interval of each region is found by substituting successive values of these truncated series until the required boundary conditions are satisfied. The method is applied to a third order type two system whose sustained oscillation, when subjected to dry friction, is to be eliminated by dead-zone compensation. The system has four regions with different parameters for each region of the differential equations which are converted into Volterra integral equations of the second kind. The variables are iterated within the digital computer until a convergent solution is found for the condition of sustained oscillation. Procedures are given to determine critical values of dead-zone for various ramp rates at

which the system is stable. AUTHOR 16358

Introduction

In automatic control systems many nonlinearities are actually piece-wise linear, or can be approximated as such. It is the purpose of this paper to analyze systems of this type by the solution of their corresponding integral equations. The method becomes particularly advantageous through the use of digital computers.

Piece-wise Linear Elements

A piece-wise linear element is shown in Figure 1 where q_j, the output of the j th element, is a linear function of an input c_j within each of I different intervals. If $c_{j,i-1} \leq c_j \leq c_{j,i}$, then

$$q_j = N_j(c_j) = k_{j,i} + \gamma_{j,i} c_j, \qquad i = 1,2,\cdots, I. \tag{1}$$

where $\gamma_{j,i}$, $k_{j,i}$ = constant gain and offset respectively,

$c_{j,i-1}$ = starting point of i th interval,

$c_{j,i}$ = end point of i th interval,

j = index for a particular piece-wise linear element,

and i = index for a particular linear piece of the element.

It is seen that this representation is inadequate if the nonlinearity results from, for instance, hysteresis given in the example. However this does not affect the generality of the concept.

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The Piece-wise Linear System

Consider a piece-wise linear system as shown in Figure 2. The piece-wise linear elements present are denoted by N_j and U_j $(j = 0,1,\cdots,J)$ and the linear elements are denoted by the operators $G_j(p)$ and $H_j(p)$ where $p = \frac{d}{dt}$. The generality of the concept is maintained here also, although for clarity no elements have been inserted between c_j and c_{j+1}.

From Figure 2 the following formulae can be obtained:

$$m_J = G_J^{(-1)}(p)c_J, \tag{2a}$$

$$m_J = r - \sum_{j=0}^{J} H_j(p)q_j \ , \tag{2b}$$

$$c_j = G_j(p)m_j, \qquad (j = 0,1,\cdots,J). \tag{3}$$

The piece-wise linear elements U_j have similar characteristics to N_j. Therefore,

$$M_j = U_{j+1}(c_{j+1}) = K_{j+1,i}+\Gamma_{j+1,i}c_{j+1}, \qquad i = 1,2,\cdots I,\quad j = 0,1,\cdots J-1. \tag{4}$$

By combining equations (1) - (4) one obtains

$$\begin{aligned}[G_J^{(-1)}(p)+H_J(p)\gamma_{J,i}]c_J = r-&\sum_{j=0}^{J}H_j(p)k_{j,i}\\ -&\sum_{j=0}^{J-1}H_j(p)\gamma_{j,i}G_j(p)K_{j+1,i}\\ -&\sum_{j=0}^{J-1}H_j(p)\gamma_{j,i}G_j(p)\Gamma_{j+1,i}c_{j+1}.\end{aligned} \tag{5}$$

From equations (3) and (4)

$$c_{j+1} = \Gamma_{j+1,i}^{(-1)}\ [G_j^{(-1)}(p)c_j - K_{j+1,i}], \qquad j = 0,1,\cdots J. \tag{6}$$

Equation (6) is a recursion relation from which c_{j+1} may be reduced to a function of c_o, thus equation (6) becomes

$$c_{j+1} = - \Gamma^{(-1)}_{j+1,i} K_{j+1,i} - \sum_{\mu=1}^{j} [\prod_{v=\mu}^{j} G_v^{(-1)}(p)][\prod_{v=\mu}^{j+1} \Gamma^{(-1)}_{v,i}] K_{\mu,i}$$

$$+ [\prod_{v=0}^{j} G_v^{(-1)}(p)][\prod_{v=0}^{j+1} \Gamma^{(-1)}_{v,i}] c_o \ , \quad j = 0,1,\cdots, J-1. \quad (7)$$

If $j = J-1$ in equation (7), then c_J can be obtained in terms of c_o. Substituting equation (7) for c_J and c_{j+1} in equation (5) a linear differential equation with c_o as the dependent variable can be written in the following form:

$$[p^N + A_o p^{N-1} + \cdots + A_n p^{N-n-1} + \cdots + A_{N-2} p + A_{N-1}] c_o = f_o(t) \ . \quad (8)$$

where the coefficients A_n are functions of $\gamma_{j,i}$, $\Gamma_{j,i}$, and the parameters in $G_j(p)$ and $H_j(p)$, while the forcing function $f_o(t)$ also depends on the above parameters, as well as $k_{j,i}$, $K_{j,i}$, and the input r.

It can be shown that the general form of equation (8) will be

$$[p^N + A_o p^{N-1} + \cdots + A_n p^{N-n-1} + \cdots + A_{N-2} p + A_{N-1}] c_j = f_j(t). \quad (9)$$

where $j = 0, 1, 2, \cdots, J$.

Equations (8) and (9) are linear differential equations with constant coefficients and as such can be solved by standard methods. However, for every change in any parameter it becomes necessary to find new roots of the equations, and to determine the n arbitrary constants by solving the n x n-matrix of the nth order equation. This often is not convenient especially if the equation is of higher order. To eliminate the step of finding the roots, the equations are changed to integral equations and solved as such by an iteration procedure.

Solution of Volterra Integral Equation

Equation (9) may be transformed into a Volterra Equation[(1)*] of the second kind as follows

$$c_j(t) = - \int_0^t K(t-\lambda)c_j(\lambda)d\lambda + F_j(t) \tag{10}$$

where $c_j(t)$ is the dependent variable,

$K(t-\lambda)$ is the kernel of the integral equation,

and $F_j(t)$ is the forcing function.

It can be proved that the kernel of equation (8) is

$$K(t-\lambda) = A_o + A_1(t-\lambda) + \cdots A_{n-1}\frac{(t-\lambda)^{n-1}}{(n-1)!} \quad ,$$

and the forcing function is

$$F_j(t) = \int_o^t \frac{(t-\lambda)^{n-1}}{(n-1)!} f_j(\lambda)d\lambda + \sum_{\mu=0}^{n-1} c_{jo}^{(\mu)} \frac{t^\mu}{\mu!}$$

where the $c_{jo}^{(\mu)}$ are initial conditions. Let the solution of the integral equation be of the form[(2)]

$$c_j(t) = F_j(t) + \int_{\lambda=0}^t L(t-\lambda)F_j(\lambda)d\lambda, \tag{11}$$

then from equation (A1), $L(t-\lambda)$ is of the form

$$L(t-\lambda) = B_o + B_1\frac{(t-\lambda)}{1!} + \cdots + B_m\frac{(t-\lambda)^m}{m!} + \cdots \quad . \tag{12}$$

From Appendix A the equation which determines the coefficients B_m is

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* See references at the end of the paper.

$$\sum_{m=0}^{\infty} B_m \frac{(t-\lambda)^m}{m!} + \sum_{n=0}^{\infty} A_n \frac{(t-\lambda)^n}{n!} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_n B_m \frac{(t-\lambda)^{n+m+1}}{(n+m+1)!} = 0. \tag{13}$$

If the parameters A_n are given in equation (9), the parameters B_m are found by setting the coefficients of equal powers of $(t-\lambda)$ to zero. Thus $L(t-\lambda)$ is known and c_j is obtained from equation (11). It can be proved that $L(t-\lambda)$ is a convergent infinite series.

Computer Procedure

Although the kernel $K(t-\lambda)$ is the same for all j within a particular time interval yet $F_j(t)$ depends on a particular j and its initial condition. Equations (11) - (13) give the solutions $c_j(t)$, $(j = 0, 1, \cdots J)$ since the kernel $K(t-\lambda)$ and the forcing function $F_j(t)$ are known for each interval $c_{j,i-1} \leq c_j \leq c_{j,i}$ in Figure 1. The computation starts with a trial value of the upper limit $t = t_1$ of the integral in equation (11). For this particular time interval t_1 the computed value of $c_j(t_1)$ should be within the allowable piece-wise linear range, i.e.

$$c_{j,i-1} < c_j(t_1) < c_{j,i}, \quad j = 0,1,2,\cdots, J. \tag{14}$$

The time interval is increased until the open interval in equation (14) ceases to hold for any j, $(j=0,1\cdots J)$. Then a time interval t_2 should be found such that either $c_j(t_2) \cong c_{j,i-1}$ at its lower bound or $c_j(t_2) \cong c_{j,i}$ at its upper bound. The end conditions at $t = t_2$ should be computed so that a new forcing function $F_j(t)$ and a new kernel $K(t-\lambda)$

can be obtained. The computation procedure may be repeated until certain desired solutions are found.

Third Order Piece-wise Linear System

An example is given here for the case of a third order piece-wise linear system to be compensated by a dead-zone before the integrator as shown in Figure 3.

With an integrator in a third order type two system, it has been found that sustained oscillation occurs when the system is subjected to dry friction and zero input.(3) In order to eliminate this oscillation, it is then necessary to change from a third order to a second order system by introducing a dead-zone.(4) As the output W remains constant whenever the error e falls within the region $-\Delta < e < \Delta$, sustained oscillation can be eliminated by adequate damping in a second order system.

Let the gain K be unity and the input r be αt. From the block diagram we have

$$Q = T_d \dot{e} + e + W - K_v \dot{c} \quad , \tag{15}$$

$$\ddot{c} = Q - f(\dot{c}) = -\ddot{e}, \tag{16}$$

and

$$\dot{c} = \alpha - \dot{e} \; . \tag{17}$$

In the region of motion the friction, expressed as $f(\dot{c})$, is actually independent of the velocity $\dot{c}$ and assumes a constant value $f(\dot{c}) = \beta$ as shown in Figure 4. Combining all the above equations, the system with kinetic friction becomes

$$\ddot{e} + (K_v+T_d)\dot{e} + e = -W+K_v\alpha + \beta, \quad \text{for } \dot{e} \neq \alpha \tag{18}$$

The error, e, is used as the dependent variable here instead of the output c, to facilitate computation.

At the point of impending motion Q, e, and W have the values:

$$Q = \beta_s, \; e = e_o, \text{ and } W = W_o, \tag{19}$$

By substituting the above values into equation (15) (with $\dot{c} = 0$) one obtains,

$$W_o = \beta_s - T_d\dot{e}_o - e_o \quad . \tag{20}$$

From Figure 3 it is observed that the expression for W in equation (18) will depend upon the value of e as compared to η. A sketch of the dead-zone and integrator output is given in Figure 5. From equation (18) and Figure 5 the following equation applies to the interval $0<t<t_x$,

$$\ddot{e} + (K_v+T_d)\dot{e} + e = -\frac{1}{T_i}\int_0^t (e-\eta)dt - W_o + K_v\alpha + \beta. \tag{21}$$

$$\text{for } e>\eta \text{ and } \dot{e} \neq \alpha,$$

For the interval $t_x< t< t_y$, the error is within the dead-zone and one obtains

$$\ddot{e} + (K_v+T_d)\dot{e} + e = -W_{t_x} - W_o + K_v\alpha + \beta. \tag{22}$$

$$\text{for } |e|<\eta, \text{ and } \dot{e} = \alpha,$$

where

$$W_{t_x} = \frac{1}{T_i}\int_0^{t_x} (e-\eta)dt \quad . \tag{23}$$

For the interval $t_y<t<t_f$, the integrator starts functioning again and equation (18) becomes

$$\ddot{e} + (K_v+T_d)\dot{e} + e = -\frac{1}{T_i}\int_{t_y}^{t} (e+\eta)dt - W_{t_x} - W_o + K_v\alpha + \beta, \tag{24}$$

for $e < -\eta$ and $\dot{e} \neq \alpha$.

The series solution for a third order system by the method of integral equations is given in Appendix B. In Appendix C both the kernel and forcing function of the integral equation corresponding to this third order piece-wise linear system as given by equations (21) - (24) are determined. Determination of boundary conditions during the period of static friction is illustrated in Appendix D.

Method of Computation

The following parameters provide an example for determining critical values of the dead-zone at which the system would be stable. The coefficient K_v+T_d is set at 2.0, which is the critical value of damping for a second order system. The integration constant T_i is set at 2.5.

Computation of the infinite series obtained from the solutions of equations (21) through (24) is accomplished with the aid of an I.B.M. 650 computer. With an arbitrary value of the initial error e_o, values are assigned to the ramp rate α and the dead-zone setting η so that the system will oscillate. In order to satisfy the condition of $e = \eta$, a suitable time interval Δt is chosen and all the values of $\ddot{e}$, $\dot{e}$, e and W_{t_x} are computed at the end of the first region. If e is not greater η then the time interval is doubled and all of the above values are recalculated. This procedure continues until e is greater than η at some time, $m\,\Delta t$; then the time interval is decreased to $(m-1)\Delta t$. Here, a fraction of Δt is added to $(m-1)\Delta t$ and the computation continues as before until a time is found at which $e = \eta$. With the final values of the second region as the initial values for the third region, the computations for the third region terminates when the error rate, $\dot{e}_{t_f}$, is identical to the ramp rate, α, at which point the output of the system enters the region of no motion (stiction).

Because the parameters are assigned such that sustained oscillation will result, the acceleration at $t = t_f$ will have a value other than zero. However, since the system has come to rest, the acceleration jumps to zero at this point and a discontinuity results. At this point equation (D7) is used to compute the value of e_2 in Figure 6. If $e_2 \neq e_o$ then the cycle is repeated with a new initial error, e_2, until the error after n cycles becomes $e_{2n} = e_{2(n-1)}$. When the steady-state has been reached, the acceleration, $\ddot{e}_{t_f}$, is compared to zero. If the acceleration $\ddot{e}_{t_f} \neq 0$, then the dead-zone, η, is gradually increased until finally the values of $\ddot{e}_{t_f} = 0$, $\dot{e}_{t_f} = \alpha$ and $e_{2n} = e_{2(n-1)}$ are found. The dead-zone setting, η, which produces these values of acceleration, error rate and error is the critical setting, η_{cr}, for the particular ramp rate, beyond which no sustained oscillation is possible.

Figure 7 gives the plot of $\ddot{\epsilon}_{t_f}/\delta$ vs. η/δ. The critical points for η_{cr}/δ are the values of η/δ when $\ddot{\epsilon}_{t_f}/\delta = 0$. The truncation of the infinite series for the purpose of computation occurs whenever a term in the series is found to be zero for ten digits in the accumulator of the computer. The smaller the dead-zone the less the number of terms required.

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APPENDIX A

Solution of Integral Equation by Iterated Kernels

It can be shown that the series of iterated kernels[2] in equation (11) is

$$L(t-\lambda) = \sum_{n=1}^{\infty} (-1)^n K_n(t-\lambda), \tag{A1}$$

where the iterated kernel is defined as follows:

$$K_1(t-\lambda) = K(t-\lambda) ,$$

$$\text{and } K_n(t-\lambda_{n-1}) = \int_{\lambda=\lambda_{n-1}}^{\lambda=t} K(t-\lambda)K_{n-1}(\lambda-\lambda_{n-1})d\lambda, \; n=2,3,\cdots. \tag{A2}$$

Substituting equation (A2) into equation (A1) gives

$$L(t-\lambda) + K(t-\lambda) = - \int_{\lambda_1=\lambda}^{\lambda_1=t} K(t-\lambda_1)L(\lambda_1-\lambda)d\lambda_1 \quad . \tag{A3}$$

The kernel of equation (10) can be expanded in an infinite series of the form

$$K(t-\lambda) = A_o + A_1 \frac{(t-\lambda)}{1!} + --- + A_n \frac{(t-\lambda)^n}{n!} + \cdots . \tag{A4}$$

Substitution of Equations (12) and (A4) into (A3) yields

$$\sum_{m=0}^{\infty} B_m \frac{(t-\lambda)^m}{m!} + \sum_{n=0}^{\infty} A_n \frac{(t-\lambda)^n}{n!} = - \Phi , \tag{A5}$$

$$\text{where} \quad \Phi = \int_{\lambda_1=\lambda}^{\lambda_1=t} \sum_{n=0}^{\infty} A_n \frac{(t-\lambda_1)^n}{n!} \sum_{m=0}^{\infty} B_m \frac{(\lambda_1-\lambda)^m}{m!} d\lambda_1 \quad . \tag{A6}$$

The integral in Equation (A6) may be expressed in terms of Beta and Gamma functions. The result is equation (13).

APPENDIX B

Series Solution for a Third Order System

If the order of the original differential equation is not higher than three then the kernel in equation (A4) can be chosen so that it has parameters A_o, A_1 and A_2 which are nonzero quantities and

$$A_3 = A_4 = A_5 = \text{-----} = A_n = 0 \; . \tag{B1}$$

By expanding the terms in equation (13), and equating the coefficients of like powers of $(t-\lambda)$ to zero one obtains

$$
\begin{aligned}
&A_o + B_o = 0 \quad , \\
&A_1 + B_1 + A_oB_o = 0 \quad , \\
&A_2 + B_2 + A_oB_1 + A_1B_o = 0 \quad , \\
&B_m + A_oB_{m-1} + A_1B_{m-2} + A_2B_{m-3} = 0, \qquad m = 3,4\cdots
\end{aligned}
\tag{B2}
$$

Equations (B2) can be solved recursively for B_m.

It remains to be shown that the solution in equation (11) is an infinite series if $F(t)$ is chosen to contain finite powers of t such as

$$F(t) = b_o + b_1t + b_2\frac{t^2}{2} \; . \tag{B3}$$

Substitution of equations (12) and (B3) into (11) yields

$$\hat{e}(t) = b_o + b_1t + b_2\frac{t^2}{2} + \int_o^t \sum_{m=0}^{\infty} B_m \frac{(t-\lambda)^m}{m!}\left(b_o+b_1\lambda+b_2\frac{\lambda^2}{2}\right)d\lambda \; . \tag{B4}$$

Interchanging summation and integral signs, and integrating the terms in equation (B4) by parts, the following infinite series results:

$$\dot{e}(t) = b_o[1 + \sum_{m=0}^{\infty} B_m \frac{t^{m+1}}{(m+1)!}]$$

$$+ b_1[t + \sum_{m=0}^{\infty} B_m \frac{t^{m+2}}{(m+2)!}] + b_2[\frac{t^2}{2} + \sum_{m=0}^{\infty} B_m \frac{t^{m+3}}{(m+3)!}]. \quad (B5)$$

Integrating and collecting like powers of t, the solution for e(t) becomes,

$$e(t) = e_o + b_o t + (b_o B_o + b_1) \frac{t^2}{2!}$$

$$+ ------ + (b_o B_m + b_1 B_{m-1} + b_2 B_{m-2}) \frac{t^{m+2}}{(m+2)!} + ----- , \quad (B6)$$

$$m = 1, 2, \cdots .$$

APPENDIX C

Determination of Kernel and Forcing Function

Solving for $\dot{e}$ from equation (21), integrating term by term, and substituting $e = e_o + \int_o^t \dot{e}dt$, one obtains

$$\ddot{e} = - \int_o^t (K_v+T_d)\dot{e}dt - \int_o^t \int_o^t \dot{e}dtdt - \frac{1}{T_i} \int_o^t \int_o^t \int_o^t \dot{e}dtdtdt$$

$$- \frac{1}{T_i} \int_o^t \int_o^t (e_o - \eta)dtdt + (-W_o + K_v\alpha + \beta - e_o)t + \dot{e}_o \quad . \quad (C1)$$

The double and triple integrals can be changed to a single integral.(1) Equation (10) has the kernel

$$K(t-\lambda) = (K_v+T_d) + (t-\lambda) + \frac{1}{2T_i} (t-\lambda)^2 \quad , \quad (C2)$$

and the forcing function,

$$F(t) = e_o + (-W_o + K_v\alpha + \beta - e_o)t - \frac{1}{2T_i} (e_o - \eta)t^2 \quad . \quad (C3)$$

Comparing equation (C2) to equation (A4), the parameters become

$$A_o = K_v + T_d, \quad A_1 = 1, \quad \text{and } A_2 = \frac{1}{T_i} . \quad (C4)$$

Comparing equation (C3) to equation (B3), one obtains

$$b_o = \dot{e}_o, \quad b_1 = (-W_o + K_v\alpha + \beta - e_o), \text{ and } b_2 = -\frac{1}{T_i}(e_o - \eta). \quad (C5)$$

The solution of equation (22) is obtained in the same manner as that of equation (21). Similar analysis leads to

$$A_o = K_v + T_d, \quad A_1 = 1, \quad A_2 = 0, \quad (C6)$$

as parameters of the kernal, and

$$b_o = \dot{e}_{t_x}, \quad b_1 = (-W_{t_x} - W_o + K_v\alpha + \beta - e_{t_x}), \quad b_2 = 0, \quad (C7)$$

as parameters of the forcing function.

For the interval $t_y < t < t_f$, the variable of integration in equation (24) is changed to

$$\tau = t - t_y . \quad (C8)$$

Substituting $e = \int_0^\tau \dot{e}d\tau + e_{t_y}$, where $e_{t_y} = -\eta$, it is found that the kernel remains the same as in equation (C2). However, the parameters of the forcing function become

$$b_o = \dot{e}_{t_y}, \quad b_1 = -W_{t_x} - W_o + K_v\alpha + \beta + \eta, \text{ and } b_2 = 0. \quad (C9)$$

APPENDIX D

Boundary Conditions During Period of Static Friction

In order to determine the initial error for the next cycle it is necessary to find the torque while the output is at rest. From Figure 5 and equation (15) one obtains

$$Q = T_d\alpha + \frac{1}{T_i}\int_o^{T_1}(e+\eta)dT + \frac{1}{T_i}\int_{T_2}^{T}(e-\eta)dT + e + W_{t_f}, \quad (D1)$$

for $\dot{c} = 0$ and hence $\dot{e} = \alpha$. The term W_{t_f} is the initial value of W at T=0. Let the value of $\ddot{c}$ at T=0 be $\ddot{c}_{t_f}$ which can be computed by the method of integral equations. From equation (16) we have

$$Q = \beta + \ddot{c}_{t_f} \quad . \tag{D2}$$

The charge on the integrator is determined by equations (D1) and (D2) as

$$W_{t_f} = \beta + \ddot{c}_{t_f} - T_d\alpha - e_{t_f} \quad . \tag{D3}$$

Figure 5 shows that the error e is a linear function of time T, thus

$$e = e_{t_f} + \alpha T \quad . \tag{D4}$$

Substituting equations (D2) (D4) into equation (D1), integrating, and setting the torque Q to $\beta+\delta$ at the point of impending motion (when $T = T_3$), the acceleration $\ddot{c}_{t_f}$ becomes

$$-\ddot{c}_{t_f} = -\delta + \alpha T_3 + \frac{1}{T_i}\left[(e_{t_f} + \eta)T_1 + \alpha\frac{T_1^2}{2}\right]$$

$$+ \frac{1}{T_i}\left[(e_{t_f} - \eta)(T_3-T_2) + \frac{\alpha}{2}(T_3^2-T_2^2)\right] \quad . \tag{D5}$$

From equation (D5) and the following relations obtained from Figure 5,

$$-\eta = e_{t_f} + \alpha T_1, \quad \eta = e_{t_f} + \alpha T_2, \quad e_2 = e_{t_f} + \alpha T_3, \tag{D6}$$

it can be shown that

$$-\ddot{c}_{t_f} = -\delta - e_{t_f} - \frac{1}{2T_i\alpha}(2e_{t_f}\eta + e_{t_f}^{\ 2}) + (1 - \frac{\eta}{T_i\alpha})e_2 + \frac{1}{2T_i\alpha}e_2^2$$

$$\tag{D7}$$

The initial error, e_2, for the next cycle of motion can be computed when $\ddot{c}_{t_f}$ and e_{t_f} are known in equation (D7). Under conditions of sustained oscillation, the error e_2 is equal to the error e_o in Figure 5.

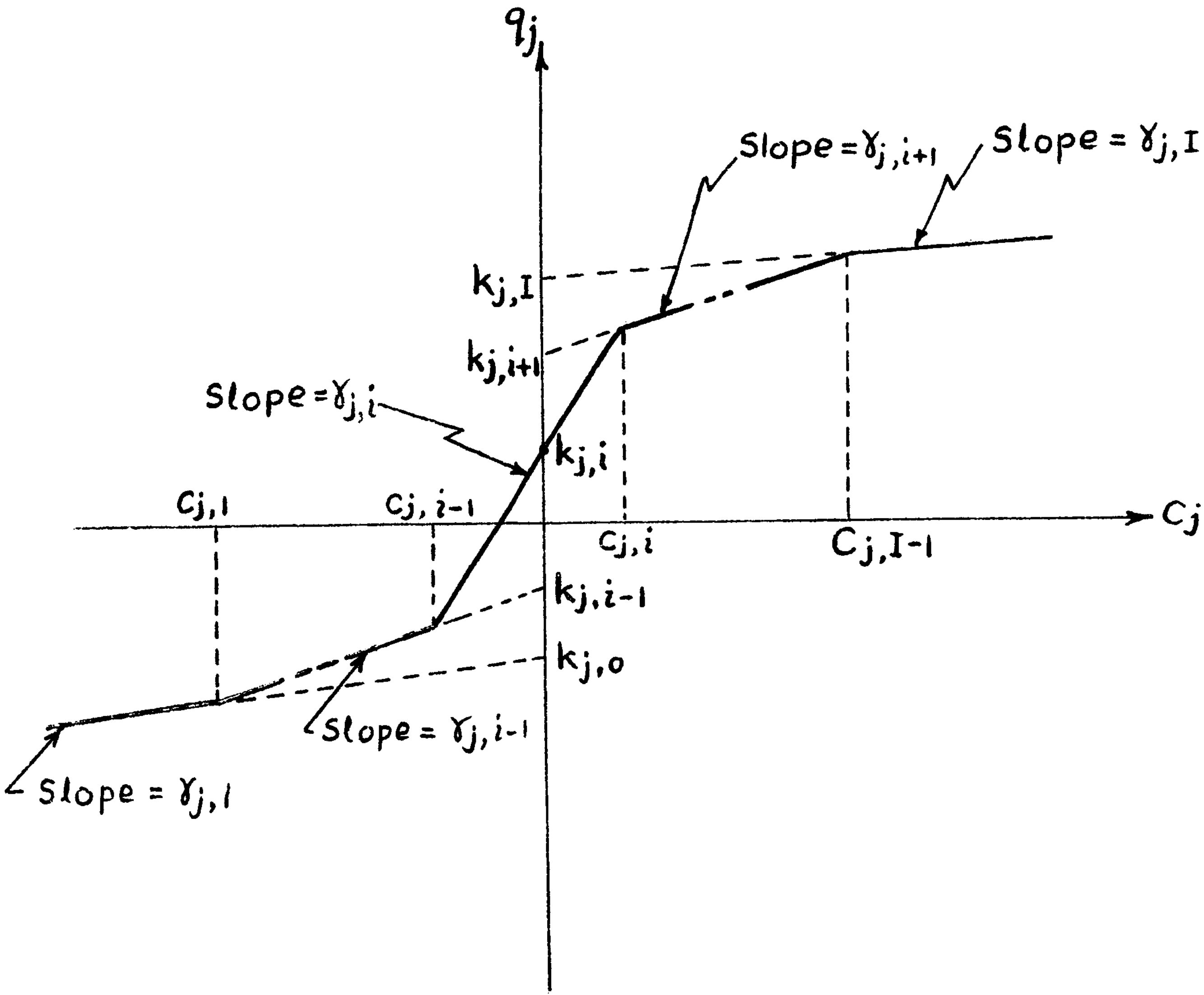

FIG. 1. PIECE-WISE LINEAR ELEMENT

(j^{th} ELEMENT)

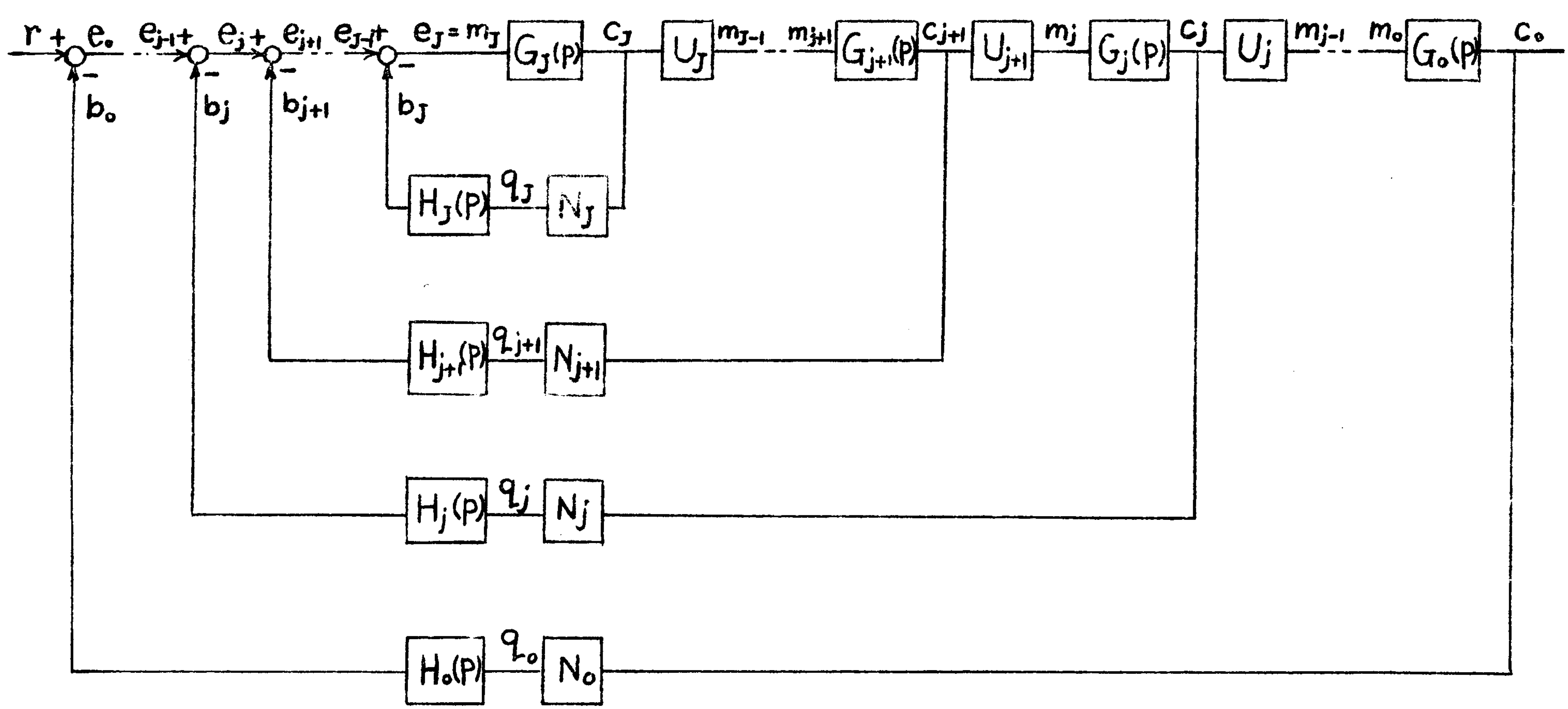

FIG. 2. PIECE-WISE LINEAR SYSTEM

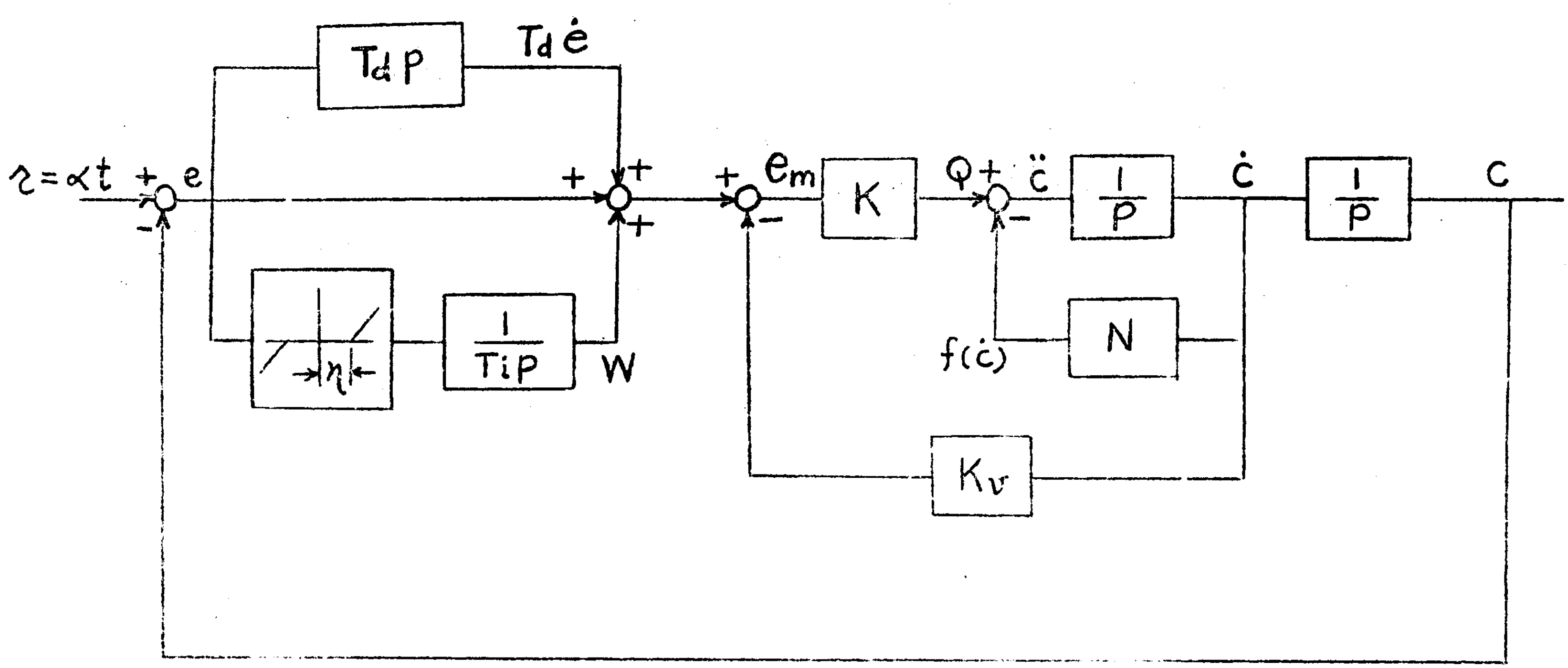

FIG. 3. THIRD ORDER SYSTEM WITH DEAD-ZONE AND DRY-FRICTION

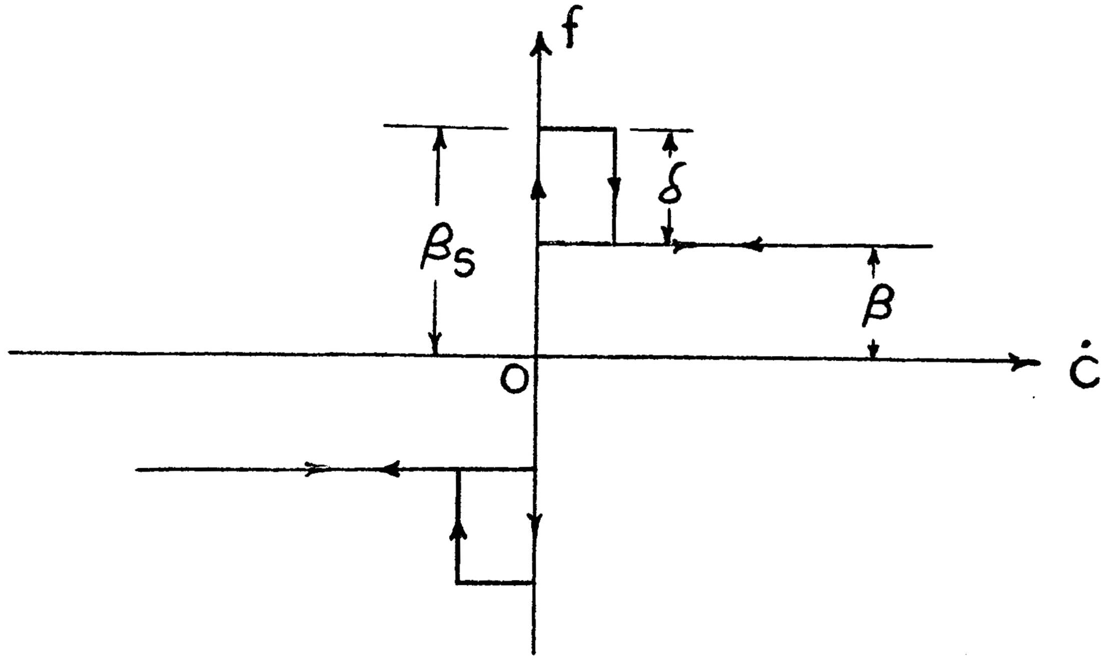

FIG. 4. FRICTION CHARACTERISTICS

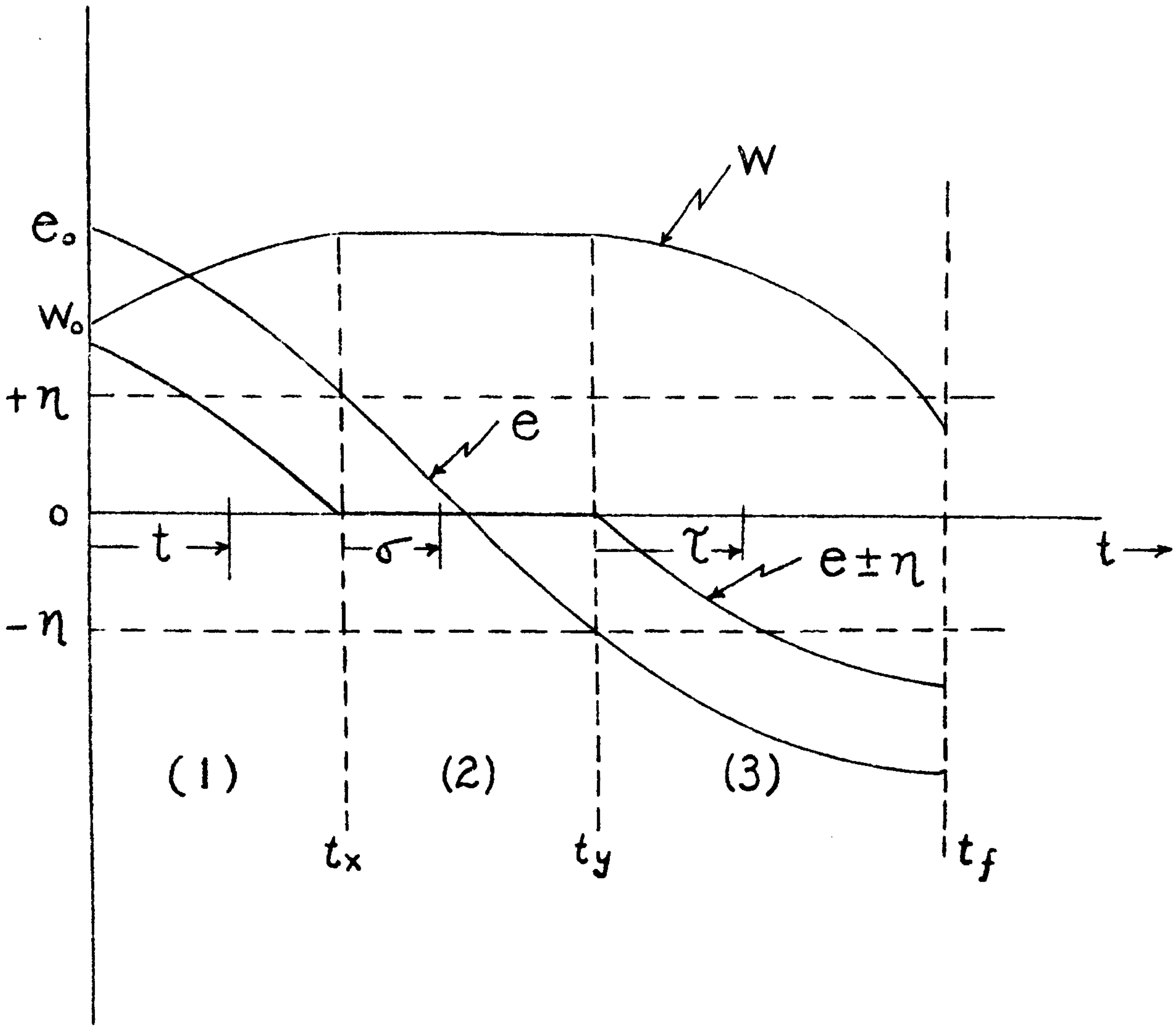

FIG. 5. DEAD-ZONE ERROR AND INTEGRATOR OUTPUT RESPONSE

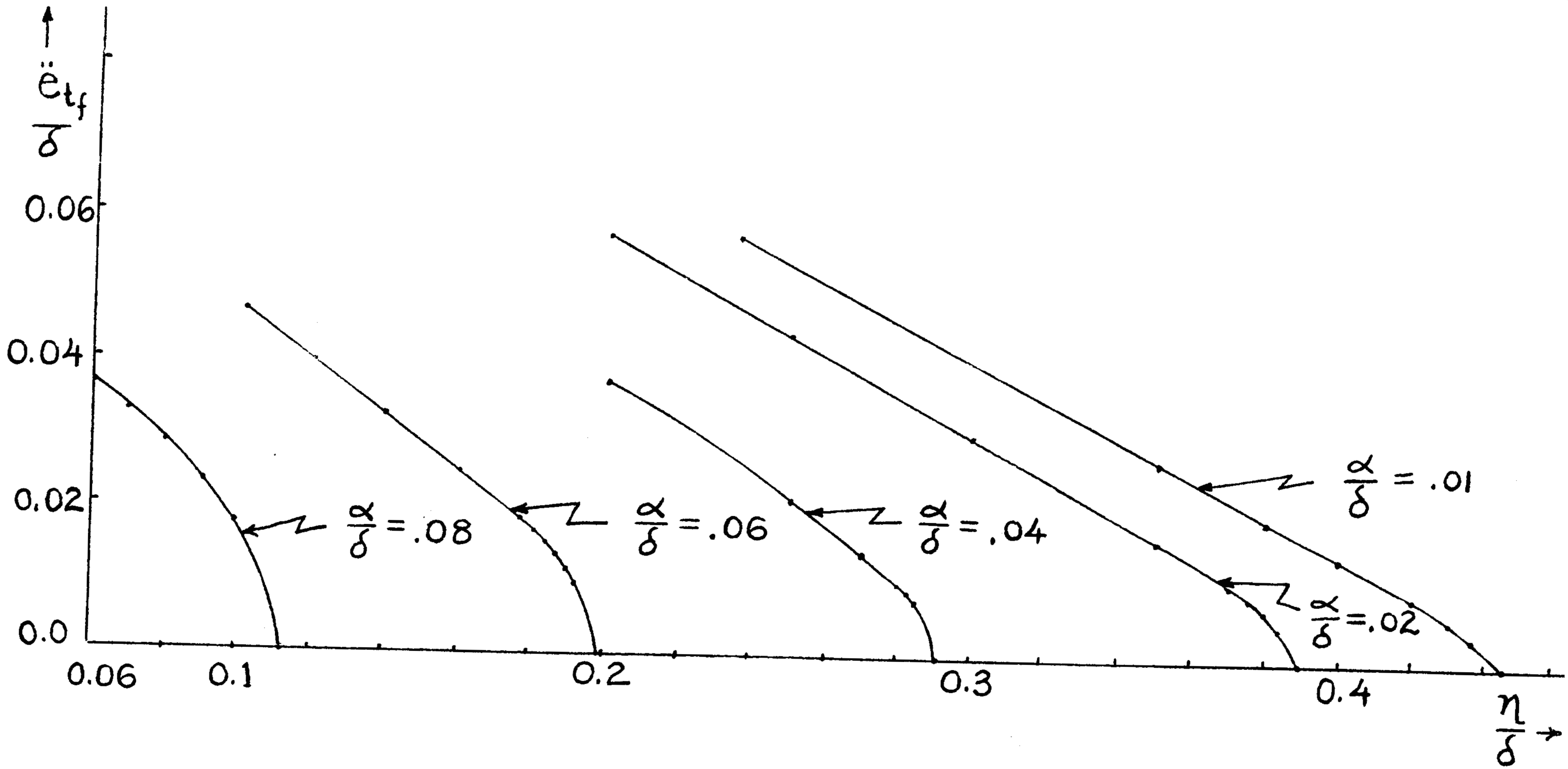

FIG. 7. FINAL ERROR ACC. VS. DEAD-ZONE FOR A THIRD ORDER SYSTEM WITH $K_v + T_d = 2$ & $T_i = 2.5$

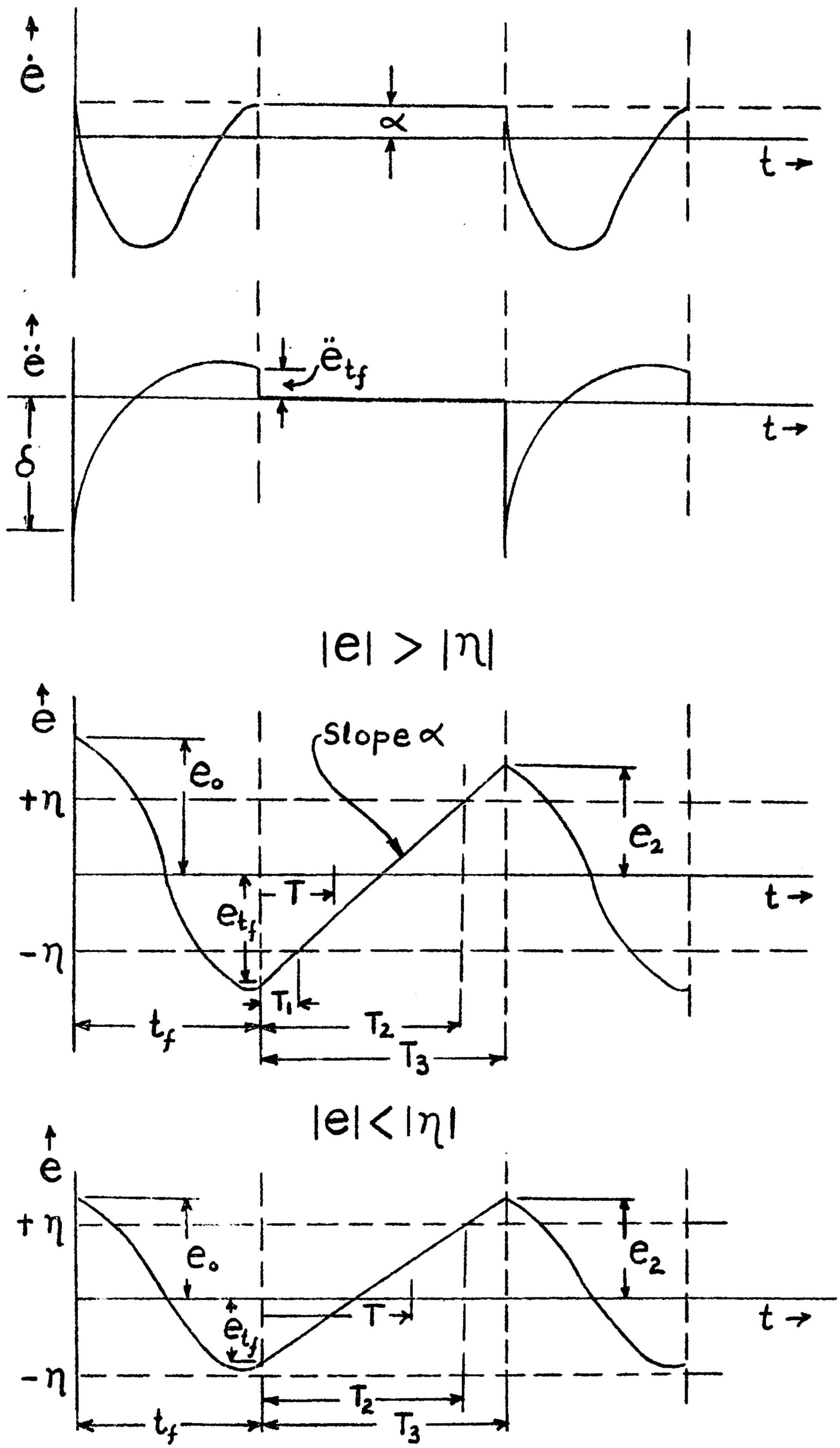

FIG. 6. SUSTAINED OSCILLATIONS